**SC531 – Lecture #03**

Let A be a subset of the sample space S. The subset A can also be called an *event*, in the sense that the proposition A is TRUE if and only if any one of the *outcomes* of subset A occurs in a given trial.

Example: Rolling a die, and A = { 1, 3, 5 }. If 3 comes up, then event A has occurred, but if 6 comes up then event A has not occurred.

The following can easily be shown:

1. P(~A) = 1 – P(A)
2. If B is a subset of A, then P(B) < P(A)
3. Extension of the third axiom to A, B, C ... [*exercise*]

A B

C

Venn’s diagrams are useful in many cases; we may imagine here that A, B, C are proper subsets of sample space S, which is not drawn.

The concept of **conditional probability** has huge practical value.

P(A|B) denotes the probability of A, *given that* B has occurred.

P(A|B) is defined by:

P(A|B) = P(A & B) / P(B) assuming P(B) ~= 0

Example: Rolling a fair die.



What is the probability that 2 is rolled, given the *additional information* that an even number has been rolled?

Answer: 1/3.

We see that the occurrence of B provides *additional information* to (re)calculate the probability of A.

Example: Rolling a pair of (fair) dice. Say the number rolled is N.

P(N = 2) = **1/36**, the unconditional probability of rolling 2.

What is the probability that 2 is rolled, *given that* the number rolled, N, is no larger than 6?

|  |  |
| --- | --- |
| Number rolled, N < 6 | Number of possible ways of rolling it |
| 2 | 1 |
| 3 | 2 |
| 4 | 3 |
| 5 | 4 |
| 6 | 5 |

P(2 is rolled | N < 6 ) = (1/36)/(15/36) = 1/15

Simple class exercise:

We roll a pair of fair dice, and it is *given that* the number rolled is 6. What is the probability that one die has rolled 5?

*In practical situations, a little additional information can change odds enormously.* *Scams are planned on such basis.*

Example: What is the probability that the boss will walk in with an umbrella, *given that* the sound of rain is clearly audible?

In making probabilistic inferences, **Bayes' rule** (or theorem) plays a hugely important role, as we shall see soon.

**Mutually exclusive events**

The occurrence of A excludes the occurrence of B, and *vice versa*.

**P(A or B) = P(A) + P(B)**  **sum rule** of mutually exclusive events

Recall and compare with the third axiom of probability. Under mutual exclusion, clearly, we have: P(A & B) = 0, P(A|B) = 0, P(B|A) = 0.

Example:

When we roll a die, P(1 or 2 is rolled) = P(1 is rolled) + P(2 is rolled)

**Mutually independent events**

The occurrence or non-occurrence of event A does not affect the occurrence or non-occurrence of event B, and *vice versa*.

In our notation: P(A|B) = P(A), and P(B|A) = P(B)

From the definitions of conditional probability and independent events, we immediately get the **product rule** of independent events:

We know that: P(A & B) = P(A|B) x P(B)

Since A, B are independent, P(A|B) = P(A). Therefore:

**P(A & B) = P(A) x P(B)** ... if A, B are independent.

In practice, we apply the above two rules all the time!

Everyday examples:

Mutually exclusive events: There is one ticket to a music program between friends A and B. So A's attendance at the program excludes B's attendance, and *vice versa*.

Independent events: A and B both have tickets to the program, and each person decides independently whether or not to attend.

Selected solved examples from Ref. #2:

1. A lot consists of 10 good articles, 4 with minor defects and 2 with major defects. Two articles are chosen from the lot at random, without replacement. Find the probabilities that: both are good, both have major defects, at least one is good, at most one is good, exactly one is good, neither has major defects, neither is good.

Probability of drawing two good articles = 10C2 / 16C2 = 3/8

Probability of drawing two with major defects = 2C2 / 16C2 = 1/120

Probability of drawing at least one good article =

Probability of drawing exactly one good article + probability of drawing both good articles = [ 10C1 x 6C1 + 10C2 ]/16C2 = 7/8

Probability of drawing at most one good article =

Probability of drawing exactly one good article + probability of drawing both defective articles = [ 10C1 x 6C1 + 6C2 ]/16C2 = 5/8

Probability of drawing exactly one good article = 10C1 x 6C1 /16C2 = 1/2

Probability of neither having major defect = 14C2 /16C2 = 91/120

Probability of both having a defect = 6C2 /16C2 = 1/8

2. A set of 6 positive and 8 negative numbers is given. Four numbers are chosen at random from the set without replacement, and multiplied together. What is the probability that the product is positive?

a. # of ways of choosing 4 positive numbers = 6C4 = 15

b. # of ways of choosing 4 negative numbers = 8C4 = 70

c. # of ways of choosing two of each = 6C2 x 8C2 = 420

Note that a, b, c are mutually exclusive events, but the two parts of c are not mutually exclusive. If fact they are independent.

Therefore the total # of ways of getting a positive product = 505

# of ways of choosing any four numbers = 14C4 = 1001

Therefore the required probability = 505/1001.

3. A pair of fair dice is rolled, and events A, B, C defined as:

A: The first die turns up an odd number,

B: The second die turns up an odd number, and

C: The sum of the two numbers rolled is odd. [why?]

Are these events mutually independent?

Firstly, note that P(A) = P(B) = P(C) = ½

Also, P(A & B) = P(A & C) = P(B & C) = ¼

So the events are pair-wise mutually independent. But the occurrence of A & B rules out the occurrence of C. That is, event (A & B) and event C are mutually exclusive.

Therefore P(A & B & C) ~= P(A) x P(B) x P(C). The LHS is in fact zero, and the RHS is 1/8. Thus the three events are not mutually independent; all three do not occur independently of each other.

Of course invariably we assume that N throws of a die, or N tosses of a coin, are mutually independent. These are actually independent trials of a random experiment.

*Early look!*

A ***random variable*** tracks the outcomes of a random trial or experiment.

Examples: Say random variable T3 represents the total obtained by rolling three dice, and random variable RFG represents the annual rainfall in Gandhinagar.

A random variable may have *discrete* or *continuous* set of values.